peak loss will be greatest in the line that has the highest $\rho_1$, because $Q$ decreases with $\rho$.

Figure 10 is a plot of the amplitude response of two first-order lines with the same $\rho$ but different $f_0$s when both structures are immersed in a dielectric medium. The physical parameters of the lines are given in the figure. The response of these lines are similar to the air dielectric lines except that the peak loss is slightly greater (as is expected) when there is dielectric present. The value of $\alpha_c$ is 0.014 at $f_0=0.4$ GHz.

Some of the above lines were built, and the amplitude measurements made on these lines were all within twelve percent of the calculated values.

## VI. Conclusion

The results show (Figs. 6 to 10) that the losses cause a dip in the amplitude response of the coupled transmission line, all-pass networks just as they do for the all-pass network composed of lumped elements. However, they also show that for practical configurations and materials, the peak amplitude loss is small for the first two periodicities. Thus, it is possible to cascade many first-order or second-order lines before the amplitude response must be equalized (for most applications).

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# Propagation Through a Twisted Medium

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*Abstract*—**An explicit solution is obtained for propagation through a uniform twisted anisotropic medium subject to the conditions that propagation is along the twist axis, and that the structure is fine. The propagation constants are altered and coupling exists between the propagation modes. A parameter is defined which indicates the tendency of the radiation to adhere to the structure of the medium. The effects at boundary discontinuities are discussed, and tapers to an isotropic medium are dealt with. The particular application of the theory to cases of polarization conversion, circular to plane, and plane to plane are discussed.**

## I. Introduction

DURING the course of development of a plastic strip polarizer for an antenna fed by a line source, it became apparent that the inclination of the strips was not a simple parameter which could be simply related to the plane of polarization. The finite thickness and cylindrical shape of the polarizer combine to require individual strips of helicoidal shape. The inclination angle of the strips varies significantly about the nominal 45°; and on looking through the polarizer, the strip structure appears to twist about the propagation axis.

Consequently, the problem of propagation through a twisted medium became of interest. Specifically, it was necessary to determine the correct strip inclination and the effect of the twist on the differential phase shift. The problem was idealized by assuming an infinitesimally fine structure, a uniform twist, and normal incidence.

Propagation through a twisted medium is a special case of a more general theory developed by Suchy.[1] Recently, a paper by van Doeren[2] has treated the twisted medium problem by means of direct computer solutions of the differential equations. The solution in explicit form for the case of a uniform twist is given in the following.

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## II. Theory of Propagation

Assume fixed axes $(x, y, z)$ and axes $(x', y', z')$ which rotate uniformly about the $z$ axis (Fig. 1). Then

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta \\ x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \quad (1)$$

where

$$\theta = pz. \quad (2)$$

$p$ is a constant which specifies a uniform right-hand twist. Relationships among vector components are also similar to (1).

The medium will be assumed to have a structure fixed in the $x'$, $y'$ axes. If the medium had zero twist, there would be two propagation constants associated with the structure equivalent to a permittivity $\kappa_2$ for polarization in which the electric vector is parallel to the $x'$ axis, and to a permittivity $\kappa_1$ for polarization in which the electric vector is parallel to the $y'$ axis; that is, $D_x' = \kappa_2 E_x'$ and $D_y' = \kappa_1 E_y'$. Then

$$\begin{aligned} D_x &= \kappa_2 E_x' \cos\theta - \kappa_1 E_y' \sin\theta \\ D_y &= \kappa_2 E_x' \sin\theta + \kappa_1 E_y' \cos\theta. \end{aligned} \quad (3)$$

The magnetic properties of the medium are assumed to be isotropic so that Maxwell's equations

$$\begin{aligned} j\omega \boldsymbol{D} &= \text{curl } \boldsymbol{H} \\ -j\mu\omega \boldsymbol{H} &= \text{curl} \boldsymbol{E} \end{aligned}$$

lead to the equation

$$\mu\omega^2 \boldsymbol{D} = \text{curl (curl } \boldsymbol{E}).$$

On the assumption that incidence is normal and along the $z$ axis, the field components must be assumed uniform in any $x$-$y$ plane. Hence all derivatives with respect to $x$ and $y$ on the right-hand side of the preceding equation vanish and the equation reduces to

$$\begin{aligned} \mu\omega^2 D_x &= -\frac{\partial^2 E_x}{\partial z^2} \\ \mu\omega^2 D_y &= -\frac{\partial^2 E_y}{\partial z^2}\,. \end{aligned} \quad (4)$$

Substituting (3) in (4)

$$\begin{aligned} &\left(\frac{\partial^2}{\partial z^2} + \beta_2^2\right)(E_x' \cos\theta) - \left(\frac{\partial^2}{\partial z^2} + \beta_1^2\right)(E_y' \sin\theta) = 0 \\ &\left(\frac{\partial^2}{\partial z^2} + \beta_2^2\right)(E_x' \sin\theta) + \left(\frac{\partial^2}{\partial z^2} + \beta_1^2\right)(E_y' \cos\theta) = 0 \end{aligned} \quad (5)$$

where

$$\begin{aligned} \beta_2^2 &= \mu\kappa_2\omega^2 \\ \beta_1^2 &= \mu\kappa_1\omega^2. \end{aligned} \quad (6)$$

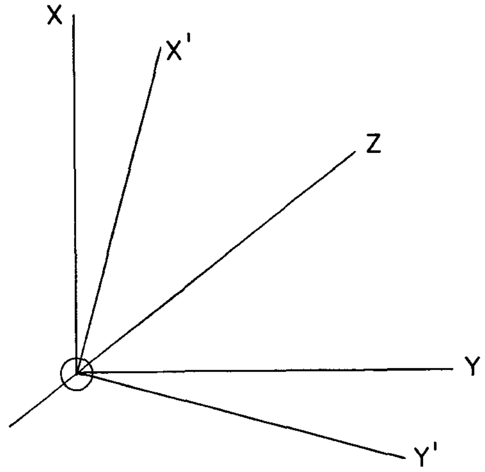


Fig. 1. The coordinate systems, $x$, $y$, $z$ are fixed axes. $x0x' = y0y' = pz$.

From the relations

$$\frac{d}{dz}(E_x' e^{j\theta}) = e^{j\theta}\left(\frac{dE_x'}{dz} + jpE_x'\right)$$

and

$$\frac{d^2}{dz^2}(E_x' e^{j\theta}) = e^{j\theta}\left(\frac{d^2E_x'}{dz^2} + 2jp\frac{dE_x'}{dz} - p^2E_x'\right)$$

and similar relations involving $E_x'e^{-j\theta}$, $E_y'e^{j\theta}$, and $E_y'e^{-j\theta}$, it is found that (5) reduces to

$$\begin{aligned} &\frac{d^2E_x'}{dz^2} + 2jp\frac{dE_x'}{dz} + (\beta_2^2 - p^2)E_x' \\ &\quad = -j\left(\frac{d^2E_y'}{dz^2} + 2jp\frac{dE_y'}{dz} + (\beta_1^2 - p^2)E_y'\right) \\ &\frac{d^2E_x'}{dz^2} - 2jp\frac{dE_x'}{dz} + (\beta_2^2 - p^2)E_x' \\ &\quad = j\left(\frac{d^2E_y'}{dz^2} - 2jp\frac{dE_y'}{dz} + (\beta_1^2 - p^2)E_y'\right). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d^2E_x'}{dz^2} + (\beta_2^2 - p^2)E_x' &= 2p\frac{dE_y'}{dz} \\ \frac{d^2E_y'}{dz^2} + (\beta_1^2 - p^2)E_y' &= -2p\frac{dE_x'}{dz} \end{aligned} \quad (7)$$

and after further manipulation

$$\frac{d^4E_x'}{dz^4} + (\beta_2^2 + \beta_1^2 + 2p^2)\frac{d^2E_x'}{dz^2} + (\beta_1^2 - p^2)(\beta_2^2 - p^2)E_x' = 0 \quad (8)$$

with a similar equation for $E_y'$. Equation (8) is satisfied by exponential functions, $e^{j\beta z}$, subject to the indicial equation

$$\beta^4 - \beta^2(\beta_2^2 + \beta_1^2 + 2p^2) + (\beta_1^2 - p^2)(\beta_2^2 - p^2) = 0. \quad (9)$$

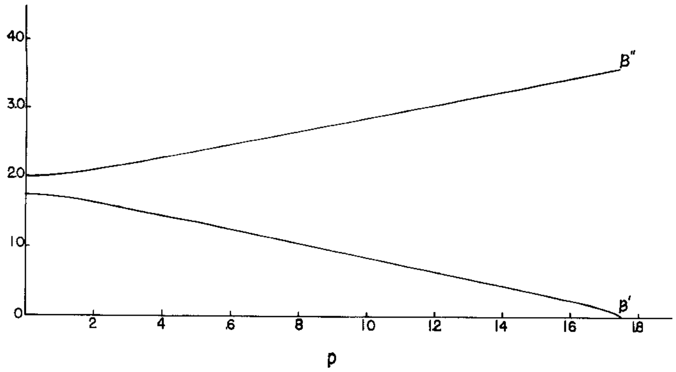


Fig. 2. Variation of $\beta''$ and $\beta'$ with $p$ for a typical numerical example. $\beta_1=1.75$, $\beta_2=2.00$. (All quantities have reciprocal units of length.)

The solutions $\beta''$ and $\beta'$ are given by

$$\beta''^2 = p^2 + \frac{\beta_2^2+\beta_1^2}{2} + \sqrt{\left(\frac{\beta_2^2-\beta_1^2}{2}\right)^2 + 2p^2(\beta_1^2+\beta_2^2)} \quad (10)$$

$$\beta'^2 = p^2 + \frac{\beta_2^2+\beta_1^2}{2} - \sqrt{\left(\frac{\beta_2^2-\beta_1^2}{2}\right)^2 + 2p^2(\beta_1^2+\beta_2^2)}. \quad (11)$$

Values of $\beta''$ and $\beta'$ for a typical case are shown plotted in Fig. 2.

For a forward traveling wave let

$$\begin{aligned} E_{x+}' &= A_+e^{-j\beta''z} + \alpha' B_+ e^{-j\beta' z} \\ jE_{y+}' &= \alpha'' A_+ e^{-j\beta''z} + B_+e^{-j\beta' z}. \end{aligned} \quad (12)$$

Substitution in (7) leads to the relations

$$\alpha' = \frac{2p\beta'}{\beta'^2-\beta_2^2+p^2} = \frac{\beta'^2-\beta_1^2+p^2}{2p\beta'} \quad (13)$$

$$\alpha'' = \frac{2p\beta''}{\beta''^2-\beta_1^2+p^2} = \frac{\beta''^2-\beta_2^2+p^2}{2p\beta''}. \quad (14)$$

Similar substitution for the backward traveling wave shows that it may be expressed

$$\begin{aligned} E_{x-}' &= A_-e^{j\beta''z} + \alpha' B_- e^{j\beta' z} \\ jE_{y-}' &= -(\alpha'' A_- e^{j\beta''z} + B_-e^{j\beta' z}). \end{aligned} \quad (15)$$

Equations (12) and (15) may be combined in the equation

$$(E') = (E_+') + (E_-') = \begin{pmatrix} 1, & \alpha' \\ -j\alpha'', & -j \end{pmatrix}\begin{pmatrix} A_+e^{-j\beta''z} \\ B_+e^{-j\beta'z} \end{pmatrix} + \begin{pmatrix} 1, & \alpha' \\ j\alpha'', & j \end{pmatrix}\begin{pmatrix} A_-e^{j\beta''z} \\ B_-e^{j\beta'z} \end{pmatrix} \quad (16)$$

in which

$$(E_+') = \begin{pmatrix} E_{x+}' \\ E_{y+}' \end{pmatrix}, \quad \text{etc.}$$

Because variation with $x$ and $y$ is zero, the magnetic field may be written

$$H_y = \frac{j}{\mu\omega}\frac{\partial E_x}{\partial z}$$

$$H_x = -\frac{j}{\mu\omega}\frac{\partial E_y}{\partial z}.$$

Substitution of the relations

$$\begin{aligned} E_x &= E_x'\cos\theta - E_y'\sin\theta \\ E_y &= E_x'\sin\theta + E_y'\cos\theta \\ \theta &= pz \end{aligned}$$

gives

$$H_x = -\frac{j}{\mu\omega}\left(\frac{\partial E_x'}{\partial z}\sin\theta + \frac{\partial E_y'}{\partial z}\cos\theta + p\cos\theta E_x' - p\sin\theta E_y'\right)$$

$$H_y = \frac{j}{\mu\omega}\left(\frac{\partial E_x'}{\partial z}\cos\theta - \frac{\partial E_y'}{\partial z}\sin\theta - p\sin\theta E_x' - p\cos\theta E_y'\right).$$

Then

$$H_x' = H_x\cos\theta + H_y\sin\theta = -\frac{j}{\mu\omega}\left(\frac{\partial E_y'}{\partial z} + pE_x'\right). \quad (17a)$$

Similarly

$$H_y' = \frac{j}{\mu\omega}\left(\frac{\partial E_x'}{\partial z} - pE_y'\right). \quad (17b)$$

Or in matrix notation

$$(H') = \frac{j}{\mu\omega}\begin{Bmatrix} -p, & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial z}, & -p \end{Bmatrix}(E'). \quad (18)$$

Substitution of (16) in (18) gives

$$(H') = \frac{j}{\mu\omega}\left\{\begin{pmatrix} -p+\alpha''\beta'', & -p\alpha'+\beta' \\ j(\alpha''p-\beta''), & j(p-\alpha'\beta') \end{pmatrix}\begin{pmatrix} A_+e^{-j\beta''z} \\ B_+e^{-j\beta'z} \end{pmatrix} + \begin{pmatrix} -p+\alpha''\beta'', & -p\alpha'+\beta' \\ -j(\alpha''p-\beta''), & -j(p-\alpha'\beta') \end{pmatrix} \cdot \begin{pmatrix} A_-e^{j\beta''z} \\ B_-e^{j\beta'z} \end{pmatrix}\right\}. \quad (19)$$

The inversion of (12) and (15) together with substitution in (19) results in

$$(H') = \frac{j}{\mu\omega}[(G+L)(E_+') + (G-L)(E_-')] \tag{20}$$

where

$$G = p\frac{\alpha'+\alpha''}{\alpha''-\alpha'}\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}$$

$$L = -j\frac{\beta'+\beta''}{\alpha''-\alpha'}\begin{pmatrix}0, & \alpha'\\ \alpha'', & 0\end{pmatrix}. \tag{21}$$

In deriving (20) and (21), use is made of four identities easily obtained from (13) and (14)

$$2p(1+\alpha'\alpha'') = (\beta''+\beta')(\alpha''+\alpha')$$
$$2p(1-\alpha'\alpha'') = (\beta''-\beta')(\alpha''-\alpha') \tag{22}$$

$$\alpha''(\beta'+\beta'')(1-\alpha'\alpha'') = (\beta''-\alpha'\alpha''\beta')(\alpha''-\alpha')$$
$$\alpha'(\beta'+\beta'')(1-\alpha'\alpha'') = -(\beta'-\alpha'\alpha''\beta'')(\alpha''-\alpha'). \tag{23}$$

## III. Transfer Representation

It is clear from the form of (12) to (20) that a coupled transmission line formalism is appropriate to the present discussion. An impedance or scattering representation could be used, but it is found convenient, instead, to use a transfer matrix representation, the purpose of which is to relate the field at one value of $z$ to that at another value of $z$. Subscripts $a$, $b$, $c$, etc. will refer to the planes $z=a$, $z=b$, $z=c$, etc., where $a<b<c$. Then

$$(E_{a+}') = \begin{pmatrix}1, & \alpha'\\ -j\alpha'', & -j\end{pmatrix}\begin{pmatrix}A_+e^{-j\beta''a}\\ B_+e^{-j\beta'a}\end{pmatrix}$$

with a similar expression for $(E_{b+}')$. The elimination of $A_+$ and $B_+$ between the two equations leads to the desired relation

$$(E_{a+}') = M(E_{b+}') \tag{24}$$

where

$$M = \frac{1}{1-\alpha'\alpha''}\cdot\begin{pmatrix}e^{j\beta''t}-\alpha'\alpha''e^{j\beta't}, & -j\alpha'(e^{j\beta''t}-e^{j\beta't})\\ -j\alpha''(e^{j\beta''t}-e^{j\beta't}), & -\alpha'\alpha''e^{j\beta''t}+e^{j\beta't}\end{pmatrix} \tag{25}$$

and $t=b-a$. It is found similarly that

$$(E_{a-}') = M^*(E_{b-}') \tag{26}$$

so that (24) and (26) may be combined

$$\begin{pmatrix}(E_{a+}')\\ (E_{a-}')\end{pmatrix} = \begin{pmatrix}M, & 0\\ 0, & M^*\end{pmatrix}\begin{pmatrix}(E_{b+}')\\ (E_{b-}')\end{pmatrix}. \tag{27}$$

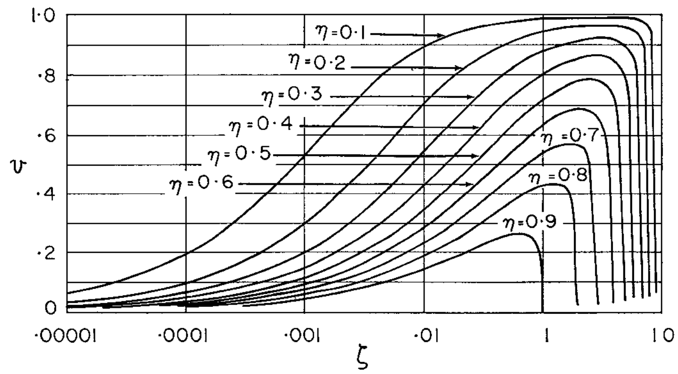


Fig. 3. $v$ as a function of the generalized parameters $\eta$ and $\zeta$. $\eta$ specifies the anisotropy and $\zeta$ the rate of twist.

Quantities $\bar{\beta}_1$ and $\bar{\beta}_2$, related directly to the properties of the medium, will be defined

$$\bar{\beta}_1^2 = \beta_1^2 - p^2; \qquad \bar{\beta}_2^2 = \beta_2^2 - p^2 \tag{28}$$

and the quantities $\epsilon$, $\phi$, $v$, and $\delta$ will be introduced

$$\epsilon = (\beta''-\beta')t = [(\bar{\beta}_2-\bar{\beta}_1)^2+4p^2]^{1/2}t \tag{29}$$

$$\phi = \frac{(\beta''+\beta')t}{2} = [(\bar{\beta}_2+\bar{\beta}_1)^2+4p^2]^{1/2}\frac{t}{2} \tag{30}$$

$$v = \sqrt{\frac{-2\alpha'\alpha''}{1-\alpha'\alpha''}} = \frac{4p}{\bar{\beta}_2+\bar{\beta}_1}\left(\frac{\bar{\beta}_1\bar{\beta}_2}{(\bar{\beta}_2-\bar{\beta}_1)^2+4p^2}\right)^{1/2} \tag{31}$$

$$\delta = \left(-\frac{\alpha'}{\alpha''}\right)^{1/4} = \left(\frac{\bar{\beta}_1}{\bar{\beta}_2}\right)^{1/4}. \tag{32}$$

Of these quantities, which are introduced because they greatly facilitate the interpretation of transfer and polarization relations, $\phi$ and $\delta$ seldom require consideration and $\epsilon$ can be regarded as a generalized differential phase shift. $v$ is an important single parameter in terms of which those special properties related to the twist of the material can be expressed. Using the new quantities, (25) becomes

$$M = \begin{pmatrix}\delta, & 0\\ 0, & \delta^{-1}\end{pmatrix}N\begin{pmatrix}\delta^{-1}, & 0\\ 0, & \delta\end{pmatrix} \tag{33}$$

where

$$N = e^{j\phi}\left\{\cos\frac{\epsilon}{2}\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix} + j\sqrt{1-v^2}\sin\frac{\epsilon}{2}\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix} + v\sin\frac{\epsilon}{2}\begin{pmatrix}0, & -1\\ 1, & 0\end{pmatrix}\right\} \tag{34}$$

$v$ is shown plotted in Fig. 3 as a function of $\zeta$ and $\eta$ which, respectively, specify the twist and anisotropy of the material and are given by

$$\zeta = \frac{2p^2}{\beta_1^2+\beta_2^2}; \qquad \eta = \frac{\kappa_2-\kappa_1}{\kappa_2+\kappa_1}. \tag{35}$$

The frequency dependence of the quantities can be obtained in a straightforward manner with the following result

$$\frac{f}{\epsilon}\frac{d\epsilon}{df} = \frac{1 - \dfrac{p^2}{\bar{\beta}_1\bar{\beta}_2}}{1 + \dfrac{4p^2}{(\bar{\beta}_2 - \bar{\beta}_1)^2}} \tag{36}$$

$$\frac{f}{\phi}\frac{d\phi}{df} = \frac{1 + \dfrac{p^2}{\bar{\beta}_1\bar{\beta}_2}}{1 + \dfrac{4p^2}{(\bar{\beta}_1 + \bar{\beta}_2)^2}} \tag{37}$$

$$\frac{f}{v}\frac{dv}{df} = -\frac{\beta_1^2\beta_2^2\left[1 - \dfrac{3}{2p^2}\left(\dfrac{1}{\beta_1^2} + \dfrac{1}{\beta_2^2}\right)\right]}{\bar{\beta}_1^2\bar{\beta}_2^2\left(1 + \dfrac{4p^2}{(\bar{\beta}_2 - \bar{\beta}_1)^2}\right)}. \tag{38}$$

Of particular interest is the fact that $d\epsilon/df = 0$ at $p = p_\epsilon$ where

$$p_\epsilon^2 = \frac{\beta_1^2\beta_2^2}{\beta_1^2 + \beta_2^2} \tag{39}$$

and $dv/df = 0$ at $p = p_v$ where

$$p_v^2 = \frac{2}{3}\frac{\beta_1^2\beta_2^2}{\beta_1^2 + \beta_2^2}. \tag{40}$$

## IV. Discontinuities and Transition Regions

The electric and magnetic fields must match at a discontintuity across the plane $z = a$ caused by a change in $\kappa_1$, $\kappa_2$, or $p$. That is, where $b = a + 0$

$$(E_{a+}') + (E_{a-}') = (E_{b+}') + (E_{b-}') \tag{41}$$

$$(H_{a+}') + (H_{a-}') = (H_{b+}') + (H_{b-}'). \tag{42}$$

The substitution of (20) in (42) gives two equations for $(E_{a+}')$ and $(E_{a-}')$, and the transfer matrix for the discontinuity boundary can be derived

$$\begin{pmatrix} E_{a+}' \\ E_{a-}' \end{pmatrix} = \begin{pmatrix} U, & V \\ V^*, & U^* \end{pmatrix}\begin{pmatrix} E_{b+}' \\ E_{b-}' \end{pmatrix} \tag{43}$$

where

$$U = \tfrac{1}{2}L_a^{-1}(G_b + L_b - G_a + L_a) = \begin{pmatrix} \tfrac{1}{2}(1 + h), & -jg \\ -jm, & \tfrac{1}{2}(1 + l) \end{pmatrix} \tag{44}$$

$$V = \tfrac{1}{2}L_a^{-1}(G_b - L_b - G_a + L_a) = \begin{pmatrix} \tfrac{1}{2}(1 - h), & -jg \\ -jm, & \tfrac{1}{2}(1 - l) \end{pmatrix} \tag{45}$$

and the quantities $h$, $l$, $g$, and $m$ are given by

$$h = \frac{(\bar{\beta}_2/\bar{\beta}_1)_b}{(\bar{\beta}_2/\bar{\beta}_1)_a} l = \frac{\left(\alpha''\dfrac{\beta'' + \beta'}{\alpha'' - \alpha'}\right)_b}{\left(\alpha''\dfrac{\beta'' + \beta'}{\alpha'' - \alpha'}\right)_a} = \frac{\left(\bar{\beta}_2\sqrt{1 + \dfrac{4p^2}{(\bar{\beta}_2 + \bar{\beta}_1)^2}}\right)_b}{\left(\bar{\beta}_2\sqrt{1 + \dfrac{4p^2}{(\bar{\beta}_2 + \bar{\beta}_1)^2}}\right)_a} \tag{46}$$

$$g = \frac{m}{(\bar{\beta}_2/\bar{\beta}_1)_a} = -\frac{1}{2}\frac{\left(p\dfrac{\alpha'' + \alpha'}{\alpha'' - \alpha'}\right)_a - \left(p\dfrac{\alpha'' + \alpha'}{\alpha'' - \alpha'}\right)_b}{\left(\alpha''\dfrac{\beta'' + \beta'}{\alpha'' - \alpha'}\right)_a} = \frac{1}{2}\frac{\left(p\dfrac{\bar{\beta}_2 - \bar{\beta}_1}{\bar{\beta}_2 + \bar{\beta}_1}\right)_b - \left(p\dfrac{\bar{\beta}_2 - \bar{\beta}_1}{\bar{\beta}_2 + \bar{\beta}_1}\right)_a}{\left(\bar{\beta}_2\sqrt{1 + \dfrac{4p^2}{(\bar{\beta}_2 + \bar{\beta}_1)^2}}\right)_a}. \tag{47}$$

The relationship for two discontinuities separated by a twist section can be immediately written

$$\begin{pmatrix} (E_{a+}') \\ (E_{a-}') \end{pmatrix} = \begin{pmatrix} U, & V \\ V^*, & U^* \end{pmatrix}_{ab}\begin{pmatrix} M, & 0 \\ 0, & M^* \end{pmatrix}_{bc}\begin{pmatrix} U, & V \\ V^*, & U^* \end{pmatrix}_{cd} \cdot \begin{pmatrix} (E_{d+}') \\ (E_{d-}') \end{pmatrix}. \tag{48}$$

An explicit expansion of (48) can be obtained if the medium at $z = d$ is the same as at $z = a$. If, in addition, $\phi = n\pi$ and $\epsilon = n'\pi$, then

$$\begin{pmatrix} E_{a+}' \\ E_{a-}' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_{d+}' \\ E_{d-}' \end{pmatrix}.$$

That is the field, irrespective of its incident polarization, emerges from the twist medium unchanged except for a rotation about the $z$ axis of angle $\theta$. A rearrangement of the equations in scattering form is straightforward and permits the direct evaluation of reflection and transmission coefficients at an interface.

The special application to a tapered or stepped transition region will be examined. In addition to the usual assumptions for a taper that reflections do not significantly accumulate, it will be supposed further that $v$ = constant, and that the increment in $p(\bar{\beta}_2 - \bar{\beta}_1/\bar{\beta}_2 + \bar{\beta}_1)$ can be neglected. Then the transfer from medium (a) to medium (b) as shown in

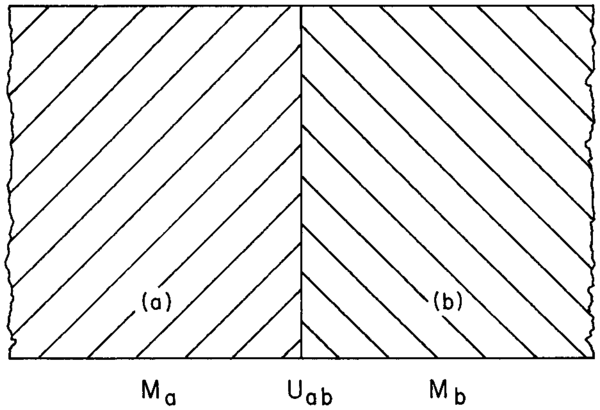


Fig. 4. The interface between two regions having different anisotropy and twist parameters.

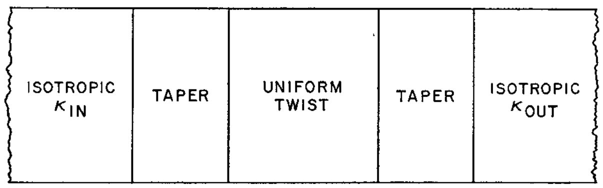


Fig. 5. The transition into and out of a twist region via tapers.

Fig. 4 will be given by the matrix product $M_aU_{ab}M_b$ in which $U_{ab}$ has zero cross terms. That is

$$M_aU_{ab}M_b = \begin{pmatrix}\delta_a, & 0\\ 0, & \delta_a^{-1}\end{pmatrix} N_a \begin{pmatrix}\delta_a^{-1}, & 0\\ 0, & \delta_a\end{pmatrix} \cdot \begin{pmatrix}\frac{1}{2}(1+h), & 0\\ 0, & \frac{1}{2}(1+l)\end{pmatrix} \cdot \begin{pmatrix}\delta_b, & 0\\ 0, & \delta_b^{-1}\end{pmatrix} N_b \begin{pmatrix}\delta_b^{-1}, & 0\\ 0, & \delta_b\end{pmatrix}.$$

In the evaluation of the inner triple product it is found that

$$\frac{1}{2}(1+h)\frac{\delta_b}{\delta_a} \approx \frac{1}{2}(1+l)\frac{\delta_b}{\delta_a} \approx \left(\frac{\bar\beta_{1b}}{\bar\beta_{1a}}\right)^{1/4}\left(\frac{\bar\beta_{2b}}{\bar\beta_{2a}}\right)^{1/4}.$$

Therefore

$$M_aU_{ab}M_b = \left(\frac{\bar\beta_{1b}}{\bar\beta_{1a}}\frac{\bar\beta_{2b}}{\bar\beta_{2a}}\right)^{1/4} \begin{pmatrix}\delta_a, & 0\\ 0, & \delta_a^{-1}\end{pmatrix} N_aN_b \begin{pmatrix}\delta_b^{-1}, & 0\\ 0, & \delta_b\end{pmatrix}.$$

It is obvious that this process is iterative through all taper regions including regions of uniform twist. Therefore, a twist medium suitably tapered from an isotropic medium of constant $\kappa_{\text{in}}$ to another isotropic medium of constant $\kappa_{\text{out}}$ as shown in Fig. 5 can be described by the transfer matrix

$$\text{transfer matrix} = \left(\frac{\kappa_{\text{out}}}{\kappa_{\text{in}}}\right)^{1/4} N \tag{49}$$

where

$$N = N_aN_bN_c \cdots. \tag{50}$$

Due to the additive property of $N$, it follows that

$$N = e^{j\phi}\left\{\cos\frac{\epsilon}{2}\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix} + j\sqrt{1-v^2}\sin\frac{\epsilon}{2}\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix} + v\sin\frac{\epsilon}{2}\begin{pmatrix}0, & -1\\ 1, & 0\end{pmatrix}\right\} \tag{34}$$

where

$$\epsilon = \epsilon_a + \epsilon_b + \epsilon_c + \cdots$$
$$\phi = \phi_a + \phi_b + \phi_c + \cdots. \tag{51}$$

It has been assumed that the tapering of the medium has been gradual and such that $v$ = constant. However, under conditions of low effective dielectric constant and low rates of twist, (49) to (51) can also be applied to a medium faced by "quarterwave" layers.

## V. Polarization Relations

The theory of the preceding sections has its most obvious application in devices for polarization conversion. It will be assumed that satisfactory transitions exist so that (49) can be applied; and as a preliminary step, the rectangular components will be related to circularly polarized components as follows

$$\begin{pmatrix}E_x'\\ E_y'\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1, & 1\\ -j, & j\end{pmatrix}\begin{pmatrix}r'\\ l'\end{pmatrix} \tag{52}$$

in which $r'$ and $l'$ are, respectively, the complex amplitudes of the right-hand and left-hand components. By the use of (52) and its inverse, it follows that

$$\begin{pmatrix}r_a'\\ l_a'\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1, & j\\ 1, & -j\end{pmatrix} N \begin{pmatrix}1, & 1\\ -j, & j\end{pmatrix}\begin{pmatrix}r_b'\\ l_b'\end{pmatrix} \tag{53}$$

from which

$$\frac{r_a'}{l_a'} = \frac{e^{-j\sigma}\dfrac{r_b'}{l_b'} + jse^{j\sigma}}{js\dfrac{r_b'}{l_b'} + e^{j\sigma}} \tag{54}$$

where

$$se^{j\sigma} = \frac{\sqrt{1-v^2}\sin\dfrac{\epsilon}{2}}{\cos\epsilon/2 + jv\sin\dfrac{\epsilon}{2}}. \tag{55}$$

Now if $r'/l' = \rho e^{2j\psi'}$, then $|\rho + 1/\rho - 1|$ is the axial ratio of the polarization ellipse, and $\psi'$ the anlge which the polarization ellipse makes with the $x'$ axis, this angle being measured to the major axis if $\rho > 1$ and to the minor axis if $\rho < 1$. Equations (54) and (55) relate these quantities at the plane $z = a$ to the corresponding quantities at the plane $z = b$ with the geometry shown in Fig. 6.

For right-hand circular polarization at $z = b$, $\rho_b = \infty$ and therefore $\rho_a e^{2j\psi'_a} = -(j/s)e^{-j\sigma}$. If $\rho_a = 1$, the input radiation is plane polarized, and the structure is a polarizer for the conversion of plane to circular polarization. It is seen from (55) that

$$2\psi_a' = -\frac{\pi}{2} + \tan^{-1}\left(v \tan \frac{\epsilon}{2}\right) \quad \text{(RH)}. \tag{56}$$

For left-hand circular $\rho_b = 0$, $\rho_a e^{2j\psi' a} = jse^{-j\sigma}$, and

$$2\psi_a' = \frac{\pi}{2} + \tan^{-1}\left(v \tan \frac{\epsilon}{2}\right) \quad \text{(LH)}. \tag{57}$$

It follows from the further relation, $s = 1$, that

$$\epsilon = \frac{\pi}{2} + \sin^{-1}\left(\frac{v^2}{1 - v^2}\right). \tag{58}$$

Thus for a polarizer structure of given $\beta_1$, $\beta_2$, and $p$, $v$ is determined by (31) and $\epsilon$ by (58). Then the necessary input inclination angle is given by (56) or (57) and the polarizer thickness by (29). Examination of (56) and (57) shows that $\psi_a'$ varies from $\mp\pi/4$ at $v = 0$, the familiar orientation for zero twist, to $\{0, \pi/2\}$ at $v^2 = 1/2$. It is seen that there is a rather marked departure from the usual polarizer relations at $v^2 = 1/2$. Furthermore, it may be shown that there is an improvement in frequency bandwidth by a factor of at least $\pi/2$. However, where a twist medium arises inadvertently due to the use of helicoidal strips as alluded to above, it is usual to find that the corrections to the usual polarizer relations are marginally significant.

If plane polarized radiation emerges at $z = b$ then $\rho_b = 1$, and if also $\rho_a = 1$, a simple rotation of plane polarization has resulted. For this condition to obtain

$$e^{2j\psi_a'} = e^{-j\sigma} \frac{e^{2j\psi_b'} + jse^{j\sigma}}{jse^{2j\psi_b'} + e^{j\sigma}}$$

making use of (55)

$$2\psi_b' = \sigma = -\tan^{-1}\left(v \tan \frac{\epsilon}{2}\right) \tag{59}$$

and

$$2\psi_a' = -\sigma = \tan^{-1}\left(v \tan \frac{\epsilon}{2}\right). \tag{60}$$

Therefore the plane of polarization has rotated through an angle $R$ given by,

$$R = \Delta\theta + \psi_b' - \psi_a' = \Delta\theta - \tan^{-1}\left(v \tan \frac{\epsilon}{2}\right). \tag{61}$$

These remarkably simple relations (which are always subject to the condition that plane polarization is converted exactly to plane polarization) are an extension of those which exist for a medium of zero twist. For the case of zero twist $R = 0$ except when $\epsilon = \pi$ in which case $R$ is indeterminate, and any rotation may be obtained. If $v$ is nonzero, a finite rotation is obtained for all thicknesses of structure and there is no indeterminacy. A numerical example of the rotation of plane polarization is shown in Fig. 7. The flat regions in $R$ have been noted by van Doeren.[2]

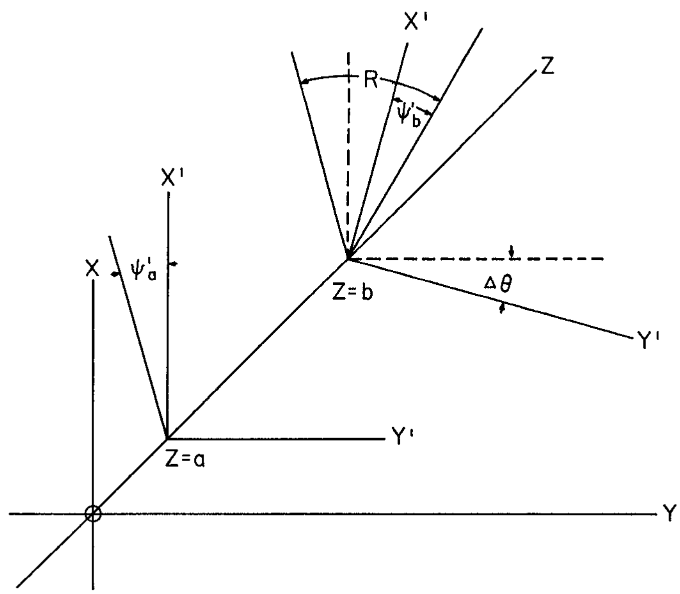


Fig. 6. The geometry for relating the polarization between the planes $z = a$ and $z = b$. The coordinate axes rotate through $\Delta\theta = p(b-a)$. $\psi_a'$ and $\psi_b'$ indicate the orientation of the polarization ellipse at $z = a$ and $z = b$. $R$ indicates the total rotation angle of the plane of polarization. For the conversion of plane polarization to plane polarization $\psi_a' = -\psi_b'$ and $R = \Delta\theta + \psi_b' - \psi_a'$.

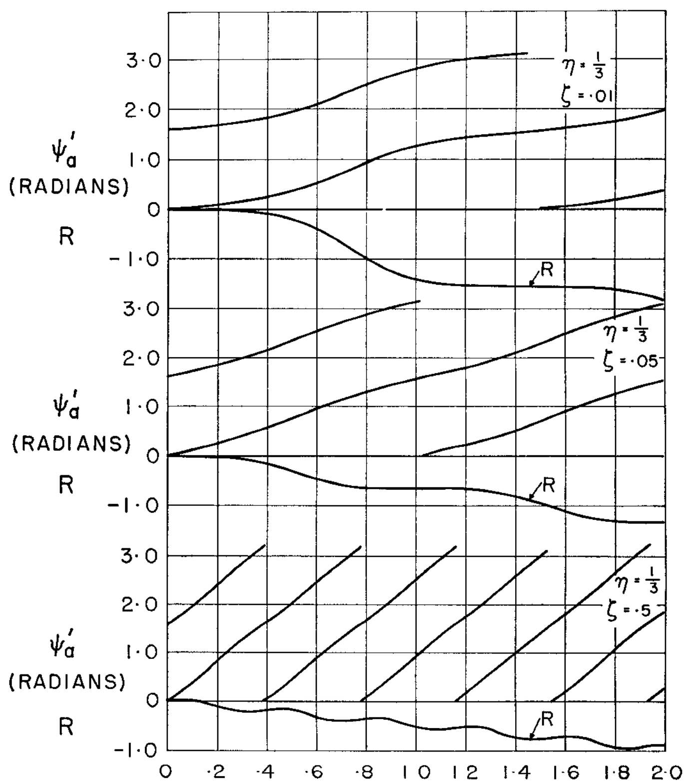


Fig. 7. $\psi_a'$ and $R$ as functions of $\tau = t/\lambda_0$ for typical numerical examples. $\psi_a'$ is the angle which the electric vector makes with the strip structure at $z = a$, while $R$ is the rotation of the polarization plane. These quantities are determined by the condition that plane polarization incident at $z = a$ emerges as plane polarization at $z = b$ where $b - a = t$.

## VI. Discussion

The main result of this investigation can be stated as follows. If the rate of twist is relatively low and the anisotropy relatively high, then the field can be thought of as attaching itself to the structure of the medium as it propagates through it, and it therefore rotates. This situation corresponds to a low value of $v$ and the conclusion stated above is evident from (34) due to the relative unimportance of the cross terms of the transfer matrix. In general, however, the twist generates polarization coupling and an alteration of propagation constants.

Discontinuities perpendicular to the direction of propagation can be handled. Computer calculation would generally be necessary, but marked simplification occurs in special cases notably for a taper region for which explicit formulas can be obtained.

A physical structure having the properties dealt with in this paper is not hard to visualize. It could consist of layers of a fabric in which warp and weft have markedly different dielectric properties, each layer being oriented at an angle with respect to the adjacent layer. Interesting speculation on this matter is contained in a recent letter by Shelton.[3] Any degree of twist per unit wavelength is possible with these structures but the range of anisotropy appears to be limited. A polarizer having a modest improvement in frequency bandwidth (corresponding to $v^2=1/2$) is feasible. The frequency independent-relations (39) and (40) can also be realized, but with a low anisotropy interesting polarization properties would require an excessive thickness of material.

Finally it should be pointed out that this paper deals with a one-dimensional problem. The lateral limitation of the geometry by means of a waveguide or other boundary would greatly complicate it.

[3] P. Shelton, "Comments on 'polarization transformation in twisted anisotropic media,'" *IEEE Trans. Microwave Theory and Techniques (Correspondence)*, vol. MTT-14, p. 579, November 1966.

# The Numerical Solution of Rectangular Waveguide Junctions and Discontinuities of Arbitrary Cross Section

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*Abstract*—**A method is described of calculating automatically the performance of junctions of rectangular waveguides including conducting cylinders of arbitrary shape. The only restriction is that the overall problem should be effectively two-dimensional, i.e., the structure be uniform in some cross section. The one basic approximation made (which could be removed) is shown to give useful results for the devices tested, viz., for various shaped irises (inductive and capacitive) and the 4-port *H*-plane junction.**

## I. Introduction

In an earlier paper [1], the authors described a method of solving the problem of the hollow waveguide of arbitrary shape, and indicated that the procedure could be applied directly to the solution of a wide range of waveguide discontinuity problems of engineering interest. The object of this paper is to describe the application and to give some typical results.

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As with the previous paper [1], the object is to enable a wide class of problems to be solvable with the one method *and the one computer program*. It should be emphasized that the technique of this paper depends on being able to calculate the cutoff frequencies of an arbitrarily shaped waveguide. Other methods have been described [2], [3] besides that used in this paper, but it is not clear from published results whether any of these is as automatic and rapid in computing.

The method can be applied directly to the analysis of a 2-, 3-, or $m$-port junction of rectangular waveguides containing arbitrarily shaped conducting structures. The waveguides may have different dimensions, but the overall structure must be uniform (i.e., have constant cross section) in one direction (either the "broad" or "narrow" transverse direction) so that the resulting boundary-value problem is effectively two-dimensional. Examples of such structures would include the conducting post or iris (of any shaped cross section) in rectangular waveguide, an offset or change of transverse dimension in the rectangular waveguide, and for $m$-port junctions the $T$, $Y$, and 4-port cross junctions. All these examples could be in the $E$ plane or $H$ plane.

The method used relies on analysis of the junction when supporting pure standing waves, as used experimentally in the "nodal-shift" or Weissfloch-Feenberg method of mea-